JOURNAL OF APPROXIMATION THEORY **86**, 179–196 (1996)
ARTICLE NO. 0061

Conjugate Jacobi Series and Conjugate Functions

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Communicated by Doron S. Lubinsky

Received July 1, 1993; accepted in revised form September 4, 1995

Conjugate Jacobi series is introduced, according to the viewpoint of Muckenhoupt and Stein, in the sense that the Jacobi harmonic function and the conjugate harmonic function satisfy the generalized Cauchy–Riemann equations. The integral form of Jacobi conjugate functions is also introduced, which is identifiable with the conjugate Jacobi series in some sense. The L_1 weak-boundedness and L_p boundedness for $1 < p < \infty$ of the conjugacy mapping are proved. The convergence of the Abel means of conjugate Jacobi series is also considered.

1. INTRODUCTION

"Conjugacy" is an important concept in classical Fourier analysis which links the study of the more fundamental properties of harmonic functions to that of analytic functions and is used to study the mean convergence of Fourier series (see Zygmund [16] and Hunt *et al.* [8]). Our particular interest will be extending the conjugacy to that related to Jacobi series in a natural and fruitful way, which will be of fundamental importance for further investigation of some deeper results known to hold in the classical case. The first work for this was done by Muckenhoupt and Stein [13], who studied some properties of conjugate ultraspherical series and the basic theory of the corresponding H^p spaces. The purpose of the present paper is to establish some basic theory of conjugate Jacobi series and conjugate functions according to the viewpoint of [13]. The study of the application of these results, especially for the corresponding H^p theory, will be done in a later paper.

The starting point of Muckenhoupt and Stein [13] is the notion of conjugacy on a Euclidean n-space as defined by Stein and Weiss [14]. They introduced the so-called generalized Cauchy–Riemann equations by considering the functions which are invariant under rotations, leaving a given axis fixed. For $\lambda > 0$, consider the set $\{P_k^\lambda(\cos\theta)\}$ of the ultraspherical polynomials, which is orthogonal over $(0, \pi)$ with respect to the measure $\sin^{2\lambda}\theta\, d\theta$. When 2λ is integral, $2\lambda = n-2$, the $P_k^\lambda(\cos\theta)$ arise in the Fourier

0021-9045/96 $18.00

analysis of functions on the surface of the sphere in Euclidean n-space. For any function f on $(0, \pi)$, its ultraspherical expansion is

$$f(\theta) \sim \sum_{n=0}^{\infty} a_n \omega_n^\lambda P_n^\lambda(\cos\theta), \tag{1.1}$$

where ω_n^λ is the orthonormalization constant (see [15, p. 81]) and $a_n = \int_0^\pi f(\varphi) P_n^\lambda(\cos\varphi) \sin^{2\lambda}\varphi \, d\varphi$. We associate to (1.1) its Poisson integral

$$U(x, y) = f(r, \theta) = \sum_{n=0}^{\infty} a_n r^n \omega_n^\lambda P_n^\lambda(\cos\theta), \qquad (x, y) = (r\cos\theta, r\sin\theta).$$

Then U satisfies the singular "Laplace equation"

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{2\lambda}{y} \cdot \frac{\partial U}{\partial y} = 0. \tag{1.2}$$

Muckenhoupt and Stein [13] defined the conjugate Poisson integral $V(x, y)$ related to U satisfying (1.2) by the generalized Cauchy–Riemann equations

$$U_y + V_x = 0, \qquad U_x - V_y - \frac{2\lambda}{y} V = 0.$$

This gives

$$V(x, y) = \tilde{f}(r, \theta) = 2\lambda \sum_{n=1}^{\infty} \frac{a_n r^n \sin\theta}{n + 2\lambda} \omega_n^\lambda P_{n-1}^{\lambda+1}(\cos\theta) \tag{1.3}$$

and leads to the generalized Hilbert transform $f \to \tilde{f}$, where

$$\tilde{f}(\theta) \sim 2\lambda \sum_{n=1}^{\infty} \frac{a_n \sin\theta}{n + 2\lambda} \omega_n^\lambda P_{n-1}^{\lambda+1}(\cos\theta).$$

In the limiting case $\lambda = 0$, we recover the usual cosine expansion because

$$\lim_{\lambda \to 0} \lambda^{-1} P_n^\lambda(\cos\theta) = (2/n)\cos n\theta$$

and its conjugate $P_{n-1}^1(\cos\theta)\sin\theta = \sin n\theta$.

In Section 2, by extending the above notion of conjugacy in a natural way we introduce the conjugate Jacobi series in the sense that the Jacobi–Poisson integral and the conjugate Jacobi–Poisson integral satisfy the generalized "Cauchy–Riemann equations (2.8)". In Section 3, we introduce the definition of Jacobi conjugate functions in integral form, i.e.,

$\tilde{f}(\theta)=\lim_{\varepsilon\to 0}\tilde{f}_\varepsilon(\theta)$, where $\tilde{f}_\varepsilon(\theta)=\int_\varepsilon^\pi \tilde{T}_\varphi f(\theta)\cdot G(\varphi)\,d\mu^{(\alpha,\beta)}(\varphi)$, which is identifiable with the conjugate Jacobi series described in Section 2 in some sense. If $\alpha=\beta=-\frac{1}{2}$, $\tilde{T}_\varphi f(\theta)$ and $G(\varphi)$ reduce to $(f(\theta-\varphi)-f(\theta+\varphi))/2$ and $(1/\pi)\operatorname{ctg}\varphi/2$ respectively, then the previous definition of $\tilde{f}$ coincides with that in the classical case. Theorem 1 in Section 4 states the L_1 weak-boundedness and L_p boundedness for $1<p<\infty$ of the conjugate mapping $f\to\tilde{f}$, whose proof is based on the exact estimates of the conjugate Jacobi–Poisson kernel. At last, in Section 5, $\tilde{f}_\varepsilon$ is used to study the convergence of the Abel means of conjugate Jacobi series.

2. CONJUGATE JACOBI SERIES

Let $P_n^{(\alpha,\beta)}(\cos\theta)$ be the Jacobi polynomial of degree n and order (α,β), $\alpha,\beta>-1$, normalized so that $P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}$. They are orthogonal and complete over $(0,\pi)$ with respect to the measure $d\mu^{(\alpha,\beta)}(\theta)=2^{\alpha+\beta+1}\sin^{2\alpha+1}\theta/2\cos^{2\beta+1}\theta/2\,d\theta$. The relation between $P_n^{(\alpha,\beta)}$and P_n^λ is

$$P_n^\lambda(\cos\theta)=\frac{\Gamma(\lambda+1/2)\,\Gamma(n+2\lambda)}{\Gamma(2\lambda)\,\Gamma(n+\lambda+1/2)}\,P_n^{(\lambda-1/2,\,\lambda-1/2)}(\cos\theta).$$

See [15, (4, 7, 1)]. Define $R_n^{(\alpha,\beta)}(\cos\theta)=P_n^{(\alpha,\beta)}(\cos\theta)/P_n^{(\alpha,\beta)}(1)$, and denote by $L_p(\alpha,\beta)$ $(1\leqslant p<\infty)$ the space of functions f for which $\|f\|_{p(\alpha,\beta)}=\{\int_0^\pi |f(\theta)|^p\,d\mu^{(\alpha,\beta)}(\theta)\}^{1/p}$ is finite, and write $|E|_{\alpha\beta}=\int_E d\mu^{(\alpha,\beta)}(\theta)$ for a set $E\subset(0,\pi)$.

For $f\in L_1(\alpha,\beta)$, its Jacobi series is

$$f(\theta)\sim\sum_{k=0}^{\infty}\hat{f}(k)\,\omega_k^{(\alpha,\beta)}R_k^{(\alpha,\beta)}(\cos\theta),\tag{2.1}$$

where

$$\hat{f}(k)=\int_0^\pi f(\varphi)\,R_k^{(\alpha,\beta)}(\cos\varphi)\,d\mu^{(\alpha,\beta)}(\varphi),$$

$$\begin{aligned}\omega_k^{(\alpha,\beta)}&=\left\{\int_0^\pi [R_k^{(\alpha,\beta)}(\cos\varphi)]^2\,d\mu^{(\alpha,\beta)}(\varphi)\right\}^{-1}\\&=2^{-\alpha-\beta-1}\frac{(2k+\alpha+\beta+1)\,\Gamma(k+\alpha+\beta+1)\,\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(k+\beta+1)\,\Gamma(k+1)}.\end{aligned}\tag{2.2}$$

The relation

$$k(k+\alpha+\beta+1)\,\omega_k^{(\alpha,\beta)}=4(\alpha+1)^2\,\omega_{k-1}^{(\alpha+1,\beta+1)}$$

will be used many times. We shall associate to (2.1) the "harmonic" function (Poisson integral)

$$f(r, \theta)=\sum_{k=0}^{\infty} r^k \hat{f}(k)\, \omega_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta), \qquad 0 \leqslant r<1, \tag{2.3}$$

and we shall write it in the form $U(x, y)=f(r, \theta)$, $(x, y)=(r\cos\theta, r\sin\theta)$. It can then be verified that $U(x, y)$ satisfies the differential equation

$$\Delta_{\alpha,\beta} U=0 \tag{2.4}$$

for (x, y) in the upper semidisc, $x^2+y^2<1$ and $y>0$, where

$$\Delta_{\alpha,\beta}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left\{\alpha+\beta+1+\frac{(\alpha-\beta)x}{(x^2+y^2)^{1/2}}\right\}\frac{1}{y}\frac{\partial}{\partial y}-\frac{\alpha-\beta}{(x^2+y^2)^{1/2}}\cdot\frac{\partial}{\partial x}.$$

The formal verification of (2.4) follows from the differential equation

$$g''(\theta)+\frac{1}{\sin\theta}\{(\alpha+\beta+1)\cos\theta+\alpha-\beta\}\, g'(\theta)+k(k+\alpha+\beta+1)\, g(\theta)=0 \tag{2.5}$$

satisfied by $g(\theta)=R_k^{(\alpha,\beta)}(\cos\theta)$ (see [15, (4, 2, 1)]).

It is not easy to see what the formal generalization of (1.3) to the Jacobi case is, since the factor $(n+2\lambda)^{-1}$ is involved. But if we use the notation $R_n^{(\lambda-1/2,\lambda-1/2)}(\cos\theta)$, then (1.3) becomes

$$\tilde{f}(r, \theta)=\frac{1}{2\lambda+1}\sum_{k=1}^{\infty} r^k k\hat{f}(k)\, \omega_k^{(\lambda-1/2,\lambda-1/2)} R_{k-1}^{(\lambda+1/2,\lambda+1/2)}(\cos\theta)\sin\theta,$$

where $\hat{f}(k)$ is defined by (2.2) with $\alpha=\beta=\lambda-1/2$. We take this as our model for the case of general $\alpha, \beta>-1$, and define the "conjugate harmonic" series (conjugate Poisson integral) of (2.3) by

$$\tilde{f}(r, \theta)=\frac{1}{2\alpha+2}\sum_{k=1}^{\infty} r^k k\hat{f}(k)\, \omega_k^{(\alpha,\beta)} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta)\sin\theta, \tag{2.6}$$

which leads to generalized Hilbert transform (conjugate series) $f\to\tilde{f}$, where

$$\tilde{f}(\theta)\sim\frac{1}{2\alpha+2}\sum_{k=1}^{\infty} k\hat{f}(k)\, \omega_k^{(\alpha,\beta)} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta)\sin\theta. \tag{2.7}$$

If we now set $V(x \cdot y) = \tilde{f}(r, \theta)$, then U and V satisfy the generalized "Cauchy–Riemann equations"

$$\begin{aligned} U_y + V_x - \frac{\alpha - \beta}{(x^2 + y^2)^{1/2}} V = 0 \\ U_x - V_y - \left\{ \alpha + \beta + 1 + \frac{(\alpha - \beta) x}{(x^2 + y^2)^{1/2}} \right\} \frac{V}{y} = 0. \end{aligned} \tag{2.8}$$

This can be verified formally by use of (2.5) and

$$R_k^{(\alpha,\beta)\prime}(t) = \frac{k(k + \alpha + \beta + 1)}{2\alpha + 2} R_{k-1}^{(\alpha+1,\beta+1)}(t). \tag{2.9}$$

See [15, (4, 21, 7)]. If we set $u(x, y) = U(x, y)$, $v(x, y) = y^{2\alpha+1}(\sqrt{x^2 + y^2} + x)^{\beta-\alpha} \cdot V(x, y)$, then we get the more symmetric system

$$\begin{aligned} v_x &= -y^{2\alpha+1}(\sqrt{x^2 + y^2} + x)^{\beta-\alpha} u_y \\ v_y &= y^{2\alpha+1}(\sqrt{x^2 + y^2} + x)^{\beta-\alpha} u_x. \end{aligned}$$

$v(x, y)$ is harmonic in the sense conjugate to $u(x, y)$, i.e.,

$$\Delta_{-\alpha-1, -\beta-1} v = 0.$$

The equations in (2.8) are first derived in Bavinck [5, Sect. 6.2] and in an unpublished note of Gasper.

For the conjugate Poisson integral (2.6), we have

$$\tilde{f}(r, \theta) = \int_0^\pi f(\varphi)\, Q(r, \theta, \varphi)\, d\mu^{(\alpha,\beta)}(\varphi), \tag{2.10}$$

where

$$Q(r, \theta, \varphi) = \frac{1}{2\alpha + 2} \sum_{k=1}^{\infty} r^k k \omega_k^{(\alpha,\beta)} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta)\, R_k^{(\alpha,\beta)}(\cos\varphi) \sin\theta$$

is the conjugate Poisson kernel. It follows from (2.9) that

$$Q(r, \theta, \varphi) = -r^{-\alpha-\beta-1} \int_0^r s^{\alpha+\beta} \frac{\partial}{\partial\theta} P^{(\alpha,\beta)}(s, \theta, \varphi)\, ds, \tag{2.11}$$

where

$$P^{(\alpha,\beta)}(s, \theta, \varphi) = \sum_{k=0}^{\infty} s^k \omega_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta)\, R_k^{(\alpha,\beta)}(\cos\varphi)$$

is the translated Poisson kernel.

For $\alpha>\beta>-1/2$, we use the product formula for Jacobi polynomials due to Koornwinder [9]

$$R_k^{(\alpha,\beta)}(\cos\theta)\, R_k^{(\alpha,\beta)}(\cos\varphi)=\int_0^\pi\int_0^1 R_k^{(\alpha,\beta)}(\cos\psi)\, dm_{\alpha\beta}(t,\xi) \qquad (2.12)$$

to get

$$P^{(\alpha,\beta)}(s,\theta,\varphi)=\int_0^\pi\int_0^1 P^{(\alpha,\beta)}(s,\psi)\, dm_{\alpha\beta}(t,\xi), \qquad (2.13)$$

where

$$\begin{aligned}\cos\psi&=2(\cos\theta/2\cos\varphi/2)^2+2(t\sin\theta/2\sin\varphi/2)^2\\&\quad+t\sin\theta\sin\varphi\cos\xi-1,\\ dm_{\alpha\beta}(t,\xi)&=c_{\alpha\beta}(1-t^2)^{\alpha-\beta-1}\, t^{2\beta+1}\sin^{2\beta}\xi\, dt\, d\xi, \qquad (2.14)\\ c_{\alpha\beta}&=\frac{2\Gamma(\alpha+1)}{\Gamma(1/2)\,\Gamma(\alpha-\beta)\,\Gamma(\beta+1/2)},\end{aligned}$$

and

$$P^{(\alpha,\beta)}(s,\psi)=\sum_{k=0}^\infty s^k\omega_k^{(\alpha,\beta)}R_k^{(\alpha,\beta)}(\cos\psi) \qquad (2.15)$$

is the Poisson kernel. It follows from (2.9) that

$$\frac{\partial}{\partial\psi}P^{(\alpha,\beta)}(s,\psi)=-2(\alpha+1)\, sP^{(\alpha+1,\beta+1)}(s,\psi)\sin\psi. \qquad (2.16)$$

Using (2.13) and (2.16) we obtain

$$\begin{aligned}Q(r,\theta,\varphi)&=2(\alpha+1)\, r^{-\alpha-\beta-1}\\&\quad\times\int_{s=0}^r\int_{\xi=0}^\pi\int_{t=0}^1 s^{\alpha+\beta+1}P^{(\alpha+1,\beta+1)}(s,\psi)\,\Omega(t,\xi)\, dm_{\alpha\beta}(t,\xi)\, ds\end{aligned} \qquad (2.17)$$

where ψ is defined by (2.14) and

$$\Omega(t,\xi)=\sin\theta\cos^2\varphi/2-t^2\sin\theta\sin^2\varphi/2-t\cos\theta\sin\varphi\cos\xi. \qquad (2.18)$$

If $\alpha > \beta = -1/2$, a limiting argument leads to the desired form for $Q(r, \theta, \varphi)$, and if $\alpha = \beta > -1/2$, the explicit formula is given by [13]:

$$Q(r, \theta, \varphi) = \frac{(2\alpha+1)(2\alpha+3)}{4\pi r^{2\alpha+1}} \int_0^r \int_0^\pi \frac{s^{2\alpha+1}(1-s^2)\, b \sin^{2\alpha} \xi}{(1-2as+s^2)^{\alpha+5/2}}\, d\xi\, ds,$$

where $a = \cos\theta\cos\varphi + \sin\theta\sin\varphi\cos\xi$, $b = 2\sin\theta\cos\varphi - 2\cos\theta\sin\varphi\cos\xi$.

3. THE DEFINITION OF CONJUGATE FUNCTION IN INTEGRAL FORM

In the trigonometric case ($\alpha = \beta = -1/2$), we can associate to a general function $f(\theta) \sim \sum a_k \cos k\theta$ its conjugate function $\tilde{f}(\theta) = \lim_{\varepsilon \to 0} \tilde{f}_\varepsilon(\theta)$, where $\tilde{f}_\varepsilon(\theta) = \int_\varepsilon^\pi \tilde{T}_\varphi f(\theta) \cdot G(\varphi)\, d\varphi$, $\tilde{T}_\varphi f(\theta) = (f(\theta-\varphi) - f(\theta+\varphi))/2$, and $G(\varphi) = (1/\pi) \operatorname{ctg} \varphi/2$. In some sense, the Fourier series of $\tilde{f}(\theta)$ is $\sum a_k \sin k\theta$. The purpose of this section is to generalize the above notions to the Jacobi case. We will find the suitable forms of $\tilde{T}_\varphi f$ and $G(\varphi)$ for general α and β and define

$$\tilde{f}_\varepsilon(\theta) = \int_\varepsilon^\pi \tilde{T}_\varphi f(\theta) \cdot G(\varphi)\, d\mu^{(\alpha,\beta)}(\varphi)$$

such that for some "good" f with expansion (2.1), the "conjugate" function $\tilde{f}(\theta) = \lim_{\varepsilon \to 0} \tilde{f}_\varepsilon(\theta)$ is of conjugate expansion (2.7).

We consider the integral representation of

$$W_k^{(\alpha,\beta)}(\theta, \varphi) = k(k+\alpha+\beta+1)\, R_{k-1}^{(a+1,\beta+1)}(\cos\theta) \times R_{k-1}^{(\alpha+1,\beta+1)}(\cos\varphi) \sin\theta \sin\varphi$$

based on $R_k^{(\alpha,\beta)}$, which is an analogue of the trigonometric formula

$$\sin k\theta \sin k\varphi = \tfrac{1}{2}(\cos k(\theta-\varphi) - \cos k(\theta+\varphi)).$$

If $\alpha = \beta > -1/2$, we use Gegenbauer's product formula (see Askey [2, (4.10)])

$$R_{k-1}^{(\alpha+1,\alpha+1)}(\cos\theta)\, R_{k-1}^{(\alpha+1,\alpha+1)}(\cos\varphi) = c_{\alpha+1} \int_0^\pi R_{k-1}^{(\alpha+1,\alpha+1)}(\cos\Psi) \sin^{2\alpha+2}\xi\, d\xi$$

and (2.9) to get, by integration by parts, that

$$W_k^{(\alpha,\alpha)}(\theta,\varphi)=\tilde{c}_\alpha\int_0^\pi R_k^{(\alpha,\alpha)}(\cos\Psi)\sin^{2\alpha}\xi\cos\xi\,d\xi,$$

where

$$\cos\Psi=\cos\theta\cos\varphi+\sin\theta\sin\varphi\cos\xi,$$
$$c_\alpha=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)\,\Gamma(1/2)},\qquad \tilde{c}_\alpha=\frac{4(\alpha+1)\,\Gamma(\alpha+2)}{\Gamma(\alpha+1/2)\,\Gamma(1/2)}. \tag{3.1}$$

If $\alpha>\beta>-1/2$, by replacing (α,β) by $(\alpha+1,\beta+1)$ in Koornwinder's product formula (2.12) and integrating the right-hand side in ξ by parts, we obtain

$$W_k^{(\alpha,\beta)}(\theta,\varphi)=\int_0^\pi\int_0^1 R_k^{(\alpha,\beta)}(\cos\psi)\,d\tilde{m}_{\alpha\beta}(t,\xi),$$

where ψ is defined by (2.14) and

$$d\tilde{m}_{\alpha\beta}(t,\xi)=\frac{8(\alpha+1)\,\Gamma(\alpha+2)}{\Gamma(1/2)\,\Gamma(\alpha-\beta)\,\Gamma(\beta+1/2)}\times(1-t^2)^{\alpha-\beta-1}\,t^{2\beta+2}\sin^{2\beta}\xi\cos\xi\,dt\,d\xi.$$

Now we define

$$\tilde{T}_\varphi f(\theta)=\frac{\tilde{c}_\alpha}{4(\alpha+1)^2}\int_0^\pi f(\Psi)\sin^{2\alpha}\xi\cos\xi\,d\xi$$

if $\alpha=\beta>-1/2$; and

$$\tilde{T}_\varphi f(\theta)=\frac{1}{4(\alpha+1)^2}\int_0^\pi\int_0^1 f(\psi)\,d\tilde{m}_{\alpha\beta}(t,\xi)$$

if $\alpha>\beta>-1/2$. Here Ψ and ψ are given by (3.1) and (2.14) respectively.

Let us recall the definition of the generalized translation operator T_φ in terms of Jacobi polynomials:

$$T_\varphi f(\theta)=c_\alpha\int_0^\pi f(\Psi)\sin^{2\alpha}\xi\,d\xi$$

if $\alpha = \beta > -1/2$; and

$$T_\varphi f(\theta) = \int_0^\pi \int_0^1 f(\psi)\, dm_{\alpha\beta}(t, \xi)$$

if $\alpha > \beta > -1/2$. An equivalent description of T_φ in the kernel form is

$$T_\varphi f(\theta) = \int_0^\pi f(\xi)\, K(\theta, \varphi, \xi)\, d\mu^{(\alpha,\beta)}(\xi),$$

where $K(\theta, \varphi, \xi)$ is Gasper's kernel. For these, see [2, 7, 9, 10, and 11].

It is easy to see that for $\alpha \geqslant \beta > -1/2$ there exists an absolute constant $M > 0$ such that

$$|\tilde{T}_\varphi f(\theta)| \leqslant M(T_\varphi |f|)(\theta). \tag{3.2}$$

Hence $\tilde{T}_\varphi$ is bounded on $L_p(\alpha, \beta)$, $1 \leqslant p < \infty$, or C, uniformly in φ, since T_φ is also (see [7, 11]). Thus for any $f \in L_1(\alpha, \beta)$, with expansion (2.1), we have

$$\tilde{T}_\varphi f(\theta) \sim \sum_{k=1}^\infty \hat{f}(k)\, \omega_{k-1}^{(\alpha+1,\beta+1)} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta) \times R_{k-1}^{(\alpha+1,\beta+1)}(\cos\varphi) \sin\theta \sin\varphi. \tag{3.3}$$

In a limiting way, the above arguments can be done and all hold for the case $\alpha > \beta = -1/2$.

Proposition 1. *Let $\alpha \geqslant \beta \geqslant -1/2$ and let $f \in X = L_p(\alpha, \beta)$, $1 \leqslant p < \infty$, or C. Then*

(i) $\tilde{T}_\varphi f(\theta) \equiv 0$ *if* $f(\theta) \equiv$ const.;

(ii) $\lim_{\varphi \to 0} \|\tilde{T}_\varphi f(\cdot)\|_X = 0$;

(iii) $\int_0^\delta |\tilde{T}_\varphi f(\theta)|^p\, d\mu^{(\alpha,\beta)}(\varphi) = o(\delta^{2\alpha+2})$, *as* $\delta \to 0$, *for almost all* $\theta \in (0, \pi)$;

(iv) *for* $f \in L_p(\alpha, \beta)$, $0 \leqslant \gamma < \min\{\alpha + \beta + 1, 2\beta + 2\}$,

$$\int_0^\pi |\tilde{T}_\varphi f(\theta)|^p\, d\mu^{(\alpha,\beta-\gamma/2)}(\varphi) \leqslant M \cos^{-\gamma}\theta \int_0^\pi |f(\varphi)|^p\, d\mu^{(\alpha,\beta)}(\varphi).$$

Proof. By the definition of $\tilde{T}_\varphi$, (i) is obvious. For $f_k(\theta) = R_k^{(\alpha,\beta)}(\cos\theta)$,

$$\tilde{T}_\varphi f_k(\theta) = \frac{k(k+\alpha+\beta+1)}{4(\alpha+1)^2} \times R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta)\, R_{k-1}^{(\alpha+1,\beta+1)}(\cos\varphi) \sin\theta \sin\varphi,$$

so that $\lim_{\varphi \to 0} \|\tilde{T}_\varphi f_k(\cdot)\|_X = 0$. Thus the completeness of $\{R_k^{(\alpha,\beta)}(\cos\theta)\}$ and the boundedness of $\tilde{T}_\varphi$ yield (ii). It follows from (3.2) and the proof of [11, Lemma 2.1] that for $0<\theta<\pi$ and $0<\delta<\min\{\theta/2, (\pi-\theta)/2\}$,

$$\int_0^\delta |\tilde{T}_\varphi f(\theta)|^p \, d\mu^{(\alpha,\beta)}(\varphi)$$

$$\leqslant M\delta^{2\alpha+1} \sin^{-2\alpha-1}\theta/2 \cos^{-2\beta-1}\theta/2 \int_{-\delta}^{\delta} |f(\theta+\varphi)|^p \, d\mu^{(\alpha,\beta)}(\theta+\varphi).$$

Using (i), an adaptation of the proof of the classical Lebesgue theorem gives the conclusion of (iii) (see [16, Vol. I, p. 65]). At last, (iv) is a direct consequence of (3.2) and Li [11, Lemma 2.2].

Next we consider the series

$$\sum_{k=1}^{\infty} \frac{2\alpha+2}{k+\alpha+\beta+1} \omega_{k-1}^{(\alpha+1,\beta+1)} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\varphi) \sin\varphi.$$

By Bavinck [4, Theorem 4.5], there exists a function, say $G(\varphi)$, which is integrable over $(0,\pi)$ with respect to the measure $\sin^{2\alpha+2}\varphi/2 \cos^{2\beta+2}\varphi/2 \, d\varphi$ and is continuous on $(0,\pi]$, such that

$$G(\varphi) \sim \sum_{k=1}^{\infty} \frac{2\alpha+2}{k+\alpha+\beta+1} \omega_{k-1}^{(\alpha+1,\beta+1)} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\varphi) \sin\varphi. \tag{3.4}$$

If f is a "good" function such that $|\hat{f}(k)|$ tends to zero so quickly, it follows from (3.3) and (3.4) that

$$\int_0^\pi \tilde{T}_\varphi f(\theta) \cdot G(\varphi) \, d\mu^{(\alpha,\beta)}(\varphi) = \sum_{k=1}^{\infty} \frac{k\hat{f}(k)}{2\alpha+2} \omega_k^{(\alpha,\beta)} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta) \sin\theta.$$

For any $f \in L_1(\alpha,\beta)$ and $0<\varepsilon<\pi$, since $G(\varphi)$ has only a singularity at $\varphi=0$, we introduce $\tilde{f}_\varepsilon$ as follows:

$$\tilde{f}_\varepsilon(\theta) = \int_\varepsilon^\pi \tilde{T}_\varphi f(\theta) \cdot G(\varphi) \, d\mu^{(\alpha,\beta)}(\varphi). \tag{3.5}$$

This is a truncated generalized Hilbert transform and plays a crucial role in the summability of conjugate Jacobi series.

An integral representation of $G(\varphi)$ can be found. Let

$$G(r,\varphi) = \sum_{k=0}^{\infty} \frac{2\alpha+2}{k+\alpha+\beta+2} r^k \omega_k^{(\alpha+1,\beta+1)} R_k^{(\alpha+1,\beta+1)}(\cos\varphi) \sin\varphi. \tag{3.6}$$

Then for $0 < r < 1$,

$$G(r, \varphi) = \frac{2\alpha + 2}{r^{\alpha+\beta+2}} \int_0^r s^{\alpha+\beta+1} P^{(\alpha+1, \beta+1)}(s, \varphi) \sin \varphi \, ds \tag{3.7}$$

where $P^{(\alpha,\beta)}(s, \varphi)$ is the Poisson kernel (2.15). Letting $r \to 1$, we have by [4, Theorem 2.4] that

$$G(\varphi) = (2\alpha + 2) \int_0^1 s^{\alpha+\beta+1} P^{(\alpha+1, \beta+1)}(s, \varphi) \sin \varphi \, ds. \tag{3.8}$$

It follows from (5.4) that

$$G(\varphi) \sim \sin^{-2\alpha-2} \varphi/2 \cos \varphi/2, \qquad 0 < \varphi < \pi.$$

It is noted that if $\alpha = \beta > -1/2$,

$$G(\varphi) = \frac{2\Gamma(\alpha + 5/2)}{\Gamma(1/2)\, \Gamma(\alpha + 1)} \int_0^1 \frac{s^{2\alpha+1}(1 - s^2) \sin \varphi}{(1 - 2s \cos \varphi + s^2)^{\alpha+5/2}} \, ds.$$

This follows from (3.8) and (5.3) by applying $\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z)$ $\Gamma(z + 1/2)$.

4. L_p THEOREM FOR CONJUGATE FUNCTIONS

This section will follow the path of [13, Sects. 7, 8] to prove the basic fact that the conjugacy mapping $f \to \tilde{f}$ is bounded on L_p for $1 < p < \infty$ and weakly bounded on L_1. Askey [1] indicated that the proof of the L_p boundedness for $1 < p < \infty$ could be done by the method in that paper (see [5, Sect. 6.2]). But his method does not work for $p = 1$. Just as the proof in [13], our proof here is based on the exact estimates for the conjugate Poisson kernel $Q(r, \theta, \varphi)$.

Set

$$P(r, \theta) = \frac{1 - r^2}{2(1 - 2r \cos \theta + r^2)} = \frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\theta,$$

$$\tilde{P}(r, \theta) = \frac{r \sin \theta}{1 - 2r \cos \theta + r^2} = \sum_{n=1}^{\infty} r^n \sin n\theta,$$

$$\rho(\theta, \varphi) = (\sin \theta/2 \sin \varphi/2)^{-\alpha-1/2} (\cos \theta/2 \cos \varphi/2)^{-\beta-1/2}.$$

LEMMA 1. *Let* $\alpha, \beta > -1$ *and* $0 \leqslant r < 1$. *Then for* $0 < \theta \leqslant \pi/2$,

(I) *if* $0 \leqslant 3\varphi \leqslant 2\theta$,

$$Q(r, \theta, \varphi) = O(\theta^{-2\alpha-2});$$

(II) *if* $0 \leqslant \varphi \leqslant 2\theta \leqslant 3\varphi$ *and* $\varphi \leqslant \frac{3}{4}\pi$, *then there are bounded functions* $u(\theta)$, *independent of* r *and* φ, *and* $v(\varphi)$, *independent of* r *and* θ, *such that*

$$Q(r, \theta, \varphi) - u(\theta)\, v(\varphi)\, \rho(\theta, \varphi)\, \tilde{P}(r, \theta - \varphi)$$
$$= O\left(\frac{1}{\theta^{2\alpha+2}} \log \frac{2\theta}{|\theta - \varphi|}\right) + O(\theta^{-2\alpha-1} P(r, \theta - \varphi));$$

(III) *if* $0 \leqslant 3\theta \leqslant 2\varphi \leqslant \frac{3}{2}\pi$,

$$Q(r, \theta, \varphi) = O(\theta \varphi^{-2\alpha-3});$$

(IV) *if* $\frac{3}{4}\pi \leqslant \varphi \leqslant \pi$,

$$Q(r, \theta, \varphi) = O(\theta).$$

Lemma 1 is a consequence of Muckenhoupt [12, Theorems (5.1), (7.1), and (8.3)]. In fact, a routine computation shows that $Q(r, \theta, \varphi)$ equals

$$2^{-\alpha-\beta-1} \rho(\theta, \varphi) \sum_{k=0}^{\infty} r^k \left(\frac{k}{k+\alpha+\beta+1}\right)^{1/2} \Phi_{k-1}^{(\alpha+1, \beta+1)}(\theta)\, \Phi_k^{(\alpha, \beta)}(\varphi),$$

where $\Phi_k^{(\alpha,\beta)}(\varphi) = 2^{(\alpha+\beta+1)/2} (\omega_k^{(\alpha,\beta)})^{1/2} R_k^{(\alpha,\beta)}(\cos \varphi) \sin^{\alpha+1/2} \varphi/2 \cos^{\beta+1/2} \varphi/2$ are the orthonormalized Jacobi polynomials. Parts (I) and (III) of Lemma 1 follow from [12, Theorem (7.1)], part (II) from [12, Theorem (8.3)], and part (IV) from [12, Theorem (5.1)].

If $\pi/2 < \theta < \pi$, the similar estimates hold for $Q(r, \theta, \varphi)$ by use of

$$Q^{(\alpha,\beta)}(r, \theta, \varphi) = -Q^{(\beta,\alpha)}(r, \pi - \theta, \pi - \varphi).$$

We will need the following estimates for Q which can be easily derived from Lemma 1 and where the symmetry on the variables θ and φ appears. In the proof of Theorem 1, the symmetry will be used (see [13, p. 12]).

LEMMA 1′. *Let* $\alpha, \beta > -1$ *and* $0 \leqslant r < 1$. *Then for* $0 < \theta, \varphi < \pi$,

(I) $Q(r, \theta, \varphi) = O(\sigma(\theta))$, $0 < \varphi < \theta/2$ *or* $\theta/2 + \pi/2 < \varphi < \pi$;

(II) $Q(r, \theta, \varphi) = O(\sigma(\varphi))$, $0 < \varphi < 2\theta - \pi$ *or* $2\theta < \varphi < \pi$;

(III) $Q(r, \theta, \varphi) - u(\theta)\, v(\varphi)\, \rho(\theta, \varphi)\, \tilde{P}(r, \theta - \varphi)$

$$= O\left(\sigma(\theta) \log + \frac{2\theta(\pi - \theta)}{|\theta - \varphi|}\right) + O(\sigma(\theta)\, P(r, \theta - \varphi) \sin \theta),$$

$$\max(\theta/2, 2\theta - \pi) < \varphi < \min(2\theta, \theta/2 + \pi/2),$$

where $\sigma(\theta) = \sin^{-2\alpha - 2} \theta/2 \cos^{-2\beta - 2} \theta/2$ *and* $u(\theta)$ *and* $v(\varphi)$ *are as in Lemma* 1 (II).

We are now in a position to state the main result on conjugate functions.

THEOREM 1. *Let* $\alpha, \beta > -1$ *and* $0 \leqslant r < 1$.

(i) *If* $f \in L_1(\alpha, \beta)$ *and* $s > 0$, *then*

$$|\{\theta \mid \tilde{f}(r, \theta) > s\}|_{\alpha\beta} \leqslant (A/s)\, \|f\|_{1(\alpha, \beta)}. \tag{4.1}$$

(ii) *If* $f \in L_p(\alpha, \beta)$, $1 < p < \infty$, *then*

$$\|\tilde{f}(r, \theta)\|_{p(\alpha, \beta)} \leqslant A_p\, \|f\|_{p(\alpha, \beta)}. \tag{4.2}$$

Moreover, there is some $\tilde{f} \in L_p(\alpha, \beta)$ *such that* $\tilde{f}(r, \theta)$ *converges in* $L_p(\alpha, \beta)$-*norm to* $\tilde{f}(\theta)$ *as* $r \to 1$ *and* $\|\tilde{f}\|_{p(\alpha, \beta)} \leqslant A_p\, \|f\|_{p(\alpha, \beta)}$. *In addition,* (2.7) *holds.*

The proof of (4.1) follows the well-understood path given in [13, pp. 40–42], except that we need to appeal to Lemma 1′. Define $N_r f(\theta) = \tilde{f}(r, \theta)$ and

$$N_r^* f(\theta) = \int_0^\pi f(\varphi)\, Q(r, \varphi, \theta)\, d\mu^{(\alpha, \beta)}(\varphi). \tag{4.3}$$

Then N_r^* is the adjoint of N_r. With these, by the Marcinkiewicz interpolation theorem (see [16, Vol. II, p. 112]), an adaptation of the argument in [13, pp. 39, 42] proves (4.2). The rest of the conclusion of part (ii) can be verified by a standard approximation method.

We recall the space B of Borel measures $d\sigma$ on $[0, \pi]$ whose norm

$$\|d\sigma\|_{(\alpha, \beta)} = 2^{\alpha + \beta + 1} \int_0^\pi \sin^{2\alpha + 1} \theta/2 \cos^{2\beta + 1} \theta/2\, |d\sigma(\theta)|$$

is finite. By the usual way, Theorem 1 implies the following corollary.

COROLLARY 1. *Let* $\alpha, \beta > -1$ *and* $d\sigma \in B$. *If* $d\sigma \sim \Sigma a_k \omega_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta)$ *and*

$$\tilde{f}(r,\theta) = \sum_{k=1}^{\infty} \frac{r^k k a_k}{2\alpha+2} \omega_k^{(\alpha,\beta)} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta)\sin\theta,$$

then for $0 < p < 1$,

$$\left\{\int_0^\pi |\tilde{f}(r,\theta)|^p \, d\mu^{(\alpha,\beta)}(\theta)\right\}^{1/p} \leqslant A_p \, \|d\sigma\|_{(\alpha,\beta)}.$$

Now we consider the inverse of the mapping $f \to \tilde{f}$. For $f \in L_1(\alpha,\beta)$,

$$f(\theta) \sim \sum_{k=1}^{\infty} \check{f}(k)\, \omega_{k-1}^{(\alpha+1,\beta+1)} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta)\sin\theta,$$

where $\check{f}(k) = \int_0^\pi f(\varphi)\, R_{k-1}^{(\alpha+1,\beta+1)}(\cos\varphi)\sin\varphi \, d\mu^{(\alpha,\beta)}(\varphi)$, define the operator S_r as follows:

$$S_r f(\theta) = (2\alpha+2) \sum_{k=1}^{\infty} \check{f}(k)\, r^k k^{-1} \omega_{k-1}^{(\alpha+1,\beta+1)} R_k^{(\alpha,\beta)}(\cos\theta).$$

It is easy to see that

$$S_r f(\theta) = \int_0^\pi f(\varphi)\, Q^*(r,\theta,\varphi)\, d\mu^{(\alpha,\beta)}(\varphi),$$

where

$$\begin{aligned} Q^*(r,\theta,\varphi) &= \sum_{k=1}^{\infty} \frac{r^k(k+\alpha+\beta+1)}{2\alpha+2} \\ &\qquad \times w_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta)\, R_{k-1}^{(\alpha+1,\beta+1)}(\cos\varphi)\sin\varphi \\ &= Q(r,\varphi,\theta) + (\alpha+\beta+1)\int_0^r s^{-1} Q(s,\varphi,\theta)\, ds. \end{aligned}$$

Thus

$$S_r = N_r^* + (\alpha+\beta+1)\int_0^r s^{-1} N_s^* \, ds, \tag{4.4}$$

where N_r^* is defined by (4.3).

Since by (4.2) and a standard duality argument $\|N_r^* f\|_{p(\alpha,\beta)} \leqslant A_p \|f\|_{p(\alpha,\beta)}$ for $1 < p < \infty$, it then follows from (4.4) and the semitrival estimate $\|N_r^* f\|_{p(\alpha,\beta)} \leqslant Ar \|f\|_{p(\alpha,\beta)}$ for $0 \leqslant r \leqslant 1/2$ that

$$\|S_r f\|_{p(\alpha,\beta)} \leqslant A_p \|f\|_{p(\alpha,\beta)}, \qquad 1 < p < \infty, \qquad 0 \leqslant r < 1. \tag{4.5}$$

COROLLARY 2. *Let* $\alpha, \beta > -1$. *If* $f \in L_p(\alpha, \beta)$, $1 < p < \infty$, *and* $\hat{f}(0) = 0$, *then*

$$B_p \|f\|_{p(\alpha,\beta)} \leqslant \|\tilde{f}\|_{p(\alpha,\beta)} \leqslant A_p \|f\|_{p(\alpha,\beta)}. \tag{4.6}$$

For any polynomial $f(\theta)$, $f(\theta) = \sum_{k=1}^{n} a_k \omega_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta)$, it follows that $\lim_{r\to 1} S_r \tilde{f}(\theta) = f(\theta)$ in $L_p(\alpha, \beta)$-norm. Then by (4.2) and (4.5), (4.6) is proved.

5. SUMMABILITY OF CONJUGATE JACOBI SERIES

In this section we will use the truncated conjugate function $\tilde{f}_\varepsilon$ defined by (3.5) to study the convergence of the Abel means $\tilde{f}(r, \theta)$ of conjugate Jacobi series.

For any $0 < r < 1$, it follows from (2.6) and (3.3) that

$$\tilde{f}(r, \theta) = \int_0^\pi \tilde{T}_\varphi f(\theta) \cdot G(r, \varphi)\, d\mu^{(\alpha,\beta)}(\varphi), \tag{5.1}$$

where $G(r, \varphi)$ is defined by (3.6).

THEOREM 2. *Let* $\alpha \geqslant \beta \geqslant -1/2$ *and* $f \in L_1(\alpha, \beta)$. *If for some* $\theta \in (0, \pi)$,

$$\int_0^\delta \tilde{T}_\varphi f(\theta)\, d\mu^{(\alpha,\beta)}(\varphi) = o(\delta^{2\alpha+2}), \qquad \text{as} \quad \delta \to 0,$$

then

$$\lim_{r\to 1} \{\tilde{f}(r, \theta) - \tilde{f}_{1-r}(\theta)\} = 0. \tag{5.2}$$

Moreover, (5.2) *holds for almost all* $\theta \in (0, \pi)$.

The proof of Theorem 2 depends on the estimates for $G(r, \theta)$.

LEMMA 2. *Let* $\alpha \geqslant \beta \geqslant -1/2$. *For* $1/2 \leqslant r < 1$, *the following estimates hold*:

(i) $G(r,\varphi) = O((1-r)^{-2\alpha-2})$, $(\partial/\partial\varphi)\, G(r, \varphi) = O((1-r)^{-2\alpha-3})$, $0 < \varphi < 1 - r$;

To prove part (ii), we use (3.7), (3.8), (5.4), and (2.16) to get that for $1-r<\varphi<\pi$,

$$\begin{aligned} G(r,\varphi)-G(\varphi) &= 2(\alpha+1)\left\{\int_r^1 + (1-r^{-\alpha-\beta-2})\int_0^r\right\} s^{\alpha+\beta+1} P^{(\alpha+1,\beta+1)}(s,\varphi)\sin\varphi\, ds \\ &= O(1)\left\{\int_r^1 + (1-r)\int_0^r\right\}(1-s)\, A(s,\varphi)^{-\alpha-5/2}\sin\varphi\, ds \\ &= O((1-r)\,\varphi^{-2\alpha-3}) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial\varphi}[G(r,\varphi)-G(\varphi)] &= 2(\alpha+1)\left\{\int_r^1 + (1-r^{-\alpha-\beta-2})\int_0^r\right\} \\ &\quad\times s^{\alpha+\beta+1}[P^{(\alpha+1,\beta+1)}(s,\varphi)\cos\varphi \\ &\quad - 2(\alpha+2)\, sP^{(\alpha+2,\beta+2)}(s,\varphi)\sin^2\varphi]\, ds \\ &= O((1-r)\,\varphi^{-2\alpha-4}). \end{aligned}$$

Now we complete the proof of Theorem 2.

It follows from (5.1) and (3.5) that $\tilde{f}(r,\theta)-\tilde{f}_{1-r}(\theta)=A+B$, where

$$A=\int_0^{1-r}\tilde{T}_\varphi f(\theta)\cdot G(r,\varphi)\, d\mu^{(\alpha,\beta)}(\varphi),$$

$$B=\int_{1-r}^{\pi}\tilde{T}_\varphi f(\theta)\cdot[G(r,\varphi)-G(\varphi)]\, d\mu^{(\alpha,\beta)}(\varphi).$$

By Lemma 2, a familiar argument leads to $A=o(1)$ and $B=o(1)$ as $r\to 1$. Proposition 1(iii) implies the last conclusion of Theorem 2.

ACKNOWLEDGMENT

The author thanks the referees for their helpful suggestions.

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To prove part (ii), we use (3.7), (3.8), (5.4), and (2.16) to get that for $1-r<\varphi<\pi$,

$$\begin{aligned}
&G(r,\varphi)-G(\varphi)\\
&\quad=2(\alpha+1)\left\{\int_r^1+(1-r^{-\alpha-\beta-2})\int_0^r\right\}s^{\alpha+\beta+1}P^{(\alpha+1,\beta+1)}(s,\varphi)\sin\varphi\,ds\\
&\quad=O(1)\left\{\int_r^1+(1-r)\int_0^r\right\}(1-s)\,A(s,\varphi)^{-\alpha-5/2}\sin\varphi\,ds\\
&\quad=O((1-r)\,\varphi^{-2\alpha-3})
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial\varphi}[G(r,\varphi)-G(\varphi)]&=2(\alpha+1)\left\{\int_r^1+(1-r^{-\alpha-\beta-2})\int_0^r\right\}\\
&\qquad\times s^{\alpha+\beta+1}[P^{(\alpha+1,\beta+1)}(s,\varphi)\cos\varphi\\
&\qquad-2(\alpha+2)\,sP^{(\alpha+2,\beta+2)}(s,\varphi)\sin^2\varphi]\,ds\\
&=O((1-r)\,\varphi^{-2\alpha-4}).
\end{aligned}$$

Now we complete the proof of Theorem 2.

It follows from (5.1) and (3.5) that $\tilde{f}(r,\theta)-\tilde{f}_{1-r}(\theta)=A+B$, where

$$A=\int_0^{1-r}\tilde{T}_\varphi f(\theta)\cdot G(r,\varphi)\,d\mu^{(\alpha,\beta)}(\varphi),$$

$$B=\int_{1-r}^{\pi}\tilde{T}_\varphi f(\theta)\cdot[G(r,\varphi)-G(\varphi)]\,d\mu^{(\alpha,\beta)}(\varphi).$$

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